Exam V
Section I
Part A — No Calculators

1. B p. 97

$y = \cos^2(2x)$

$\frac{dy}{dx} = 2\cos(2x)(-\sin(2x)) \cdot 2 = -4\sin(2x)\cos(2x)$

2. C p. 97

I.	The units on the axes are equal. At (2,2), the slope $\approx$ 1.	True
II.	As y-coordinates are approaching 8 (from above and below), the slope lines are flattening out.	True
III.	For a given value of x, the slopes vary at different heights.	False

3. C p. 98

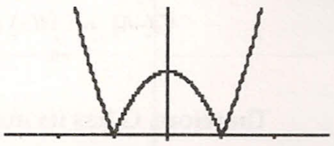
$y = \ln\sqrt{x} = \ln x^{1/2} = \frac{1}{2}\ln x$

Then $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x}$

At $(e^2, 1)$, $\frac{dy}{dx} = \frac{1}{2e^2}$

4. C p. 98

$f(x) = \left|x^2 - 1\right|$ is not differentiable at $x = \pm 1$.
The graph has sharp corners there.



$f(x) = \sqrt{x^2 - 1}$ is not continuous for x between -1 and 1 because it is undefined there.

$f(x) = \sqrt{x^2 + 1}$ is differentiable for all real x; $f'(x) = \frac{2x}{2\sqrt{x^2+1}}$. Hence the function is continuous for all real x.

$f(x) = \frac{1}{x^2 - 1}$ is not continuous at $x = \pm 1$ since the function is undefined there.

The third function (C) is both continuous and differentiable.

5. D p. 98

The slope of the line though (9,3) and (1,1) is $m = \frac{3-1}{9-1} = \frac{1}{4}$.

Since $y = \sqrt{x}$, we have $y' = \frac{1}{2\sqrt{x}}$.

Since the tangent line is to have slope m, we have $\frac{1}{2\sqrt{x}} = \frac{1}{4} \Rightarrow x = 4$.

Copyright 2006 Venture Publishing

6. A p. 99

$f(x) = \frac{x^4}{2} - \frac{x^5}{10}$

$f'(x) = 2x^3 - \frac{1}{2}x^4.$

To maximize $f'(x)$, take $f''(x) = 6x^2 - 2x^3 = 2x^2(3-x)$.

$f'(x)$ is an upside-down quartic, so has a maximum. Critical numbers for $f'(x)$ are the zeros of $f''(x)$; they are $x = 0$ and $x = 3$. $f'(x)$ is increasing on either side of $x = 0$, so that is **NOT** the maximum. $f''(x) > 0$ when $x < 3$ and $f''(x) < 0$ when $x > 3$. Therefore, f' attains its maximum at $x = 3$.

---

7. E p. 99

$a(t) = 4 - 6t$

$v(t) = \int a(t)\,dt = 4t - 3t^2 + C$

$v(0) = 20 \Rightarrow C = 20 \Rightarrow v(t) = 4t - 3t^2 + 20$

$s(t) = \int v(t)\,dt = 2t^2 - t^3 + 20t + D$

$s(3) - s(1) = (18 - 27 + 60 + D) - (2 - 1 + 20 + D) = (51 + D) - (21 + D) = 30$

---

8. D p. 99

$$\lim_{x\to1} \frac{\sqrt{x+3} - 2}{1-x} = \lim_{x\to1} \frac{[\sqrt{x+3} - 2][\sqrt{x+3} + 2]}{(1-x)[\sqrt{x+3} + 2]}$$

$$= \lim_{x\to1} \frac{(x+3) - 4}{(1-x)[\sqrt{x+3} + 2]}$$

$$= \lim_{x\to1} \frac{x-1}{(1-x)[\sqrt{x+3} + 2]}$$

$$= \lim_{x\to1} \frac{-1}{\sqrt{x+3} + 2} = -\frac{1}{4} = -0.25$$

---

9. A p. 100

To achieve continuity at $x = 1$ (the only place in question), we need $\lim_{x\to1} f(x) = f(1)$.

But $\lim_{x\to1} f(x) = \lim_{x\to1} \frac{x^2 - 2x + 1}{x-1} = \lim_{x\to1} (x-1) = 0$ while $f(1) = k$.

Therefore $k = 0$.

---

10. A p. 100

The average value of $f(x)$ on $[a, b]$ is defined to be $\frac{1}{b-a}\int_a^b f(x)\,dx$. Therefore

$$M = \frac{1}{\frac{1}{2} - 0} \cdot \int_0^{1/2} (e^{2x} + 1)\,dx = 2\left[\frac{1}{2}e^{2x} + x\right]_0^{1/2} = 2\left[\left(\frac{1}{2}e + \frac{1}{2}\right) - \left(\frac{1}{2} + 0\right)\right] = e$$

Copyright 2006 Venture Publishing

11. B p. 100

$$v(t) = \frac{e^t}{t}$$

$$v'(t) = \frac{te^t - e^t}{t^2} = \frac{e^t(t-1)}{t^2}$$, so the only critical number is $t = 1$.

If $0 < t < 1$, then $v'(t) < 0$, so v is decreasing.
If $t > 1$, then $v'(t) > 0$, so v is increasing.
Thus v achieves its minimum value at $t = 1$.

12. D p. 101

$$\int \frac{4x}{1+x^2}\,dx = 2\int \frac{2x}{1+x^2}\,dx = 2\ln(1+x^2) + C$$

(Note that $1 + x^2$ is always positive-valued.)

13. D p. 101

$f(x) = x^4 + ax^2 + b$	$f(0) = 1$	$\Rightarrow$	$1 = b$
$f'(x) = 4x^3 + 2ax$	$f'(0) = 0$	$\Rightarrow$	$0 = 0$
$f''(x) = 12x^2 + 2a$	$f''(1) = 0$	$\Rightarrow$	$0 = 12 + 2a$

Thus $a = -6$; $b = 1$.

14. A p. 101

$$\int_1^2 \frac{x^2 - x}{x^3}\,dx = \int_1^2 (x^{-1} - x^{-2})\,dx = \left[\ln|x| + \frac{1}{x}\right]_1^2$$

$$= \left(\ln 2 + \frac{1}{2}\right) - (\ln 1 + 1) = \ln 2 - \frac{1}{2}$$

15. C p. 102

Let x denote the edge of the cube. Then

$V = x^3$ $\qquad$ $SA = 6x^2$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$ $\qquad$ When $SA = 150$, then $x = 5$.

We are given that $\frac{dx}{dt} = 0.2$. Hence $\frac{dV}{dt} = 3 \cdot 5^2 \cdot (0.2) = 15$

Copyright 2006 Venture Publishing

16. B p. 102

$$\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{1+x^2}} = \int_0^{\sqrt{3}} \frac{2x}{2\sqrt{1+x^2}}\,dx = \sqrt{1+x^2}\Big]_0^{\sqrt{3}} = 2 - 1 = 1$$

This can also be done with a formal substitution.
Let $u = x^2 + 1$, so that $du = 2x\,dx$, and $x\,dx = \frac{1}{2}du$.
To change the limits of integration, we note that $x = 0 \Rightarrow u = 1$
and $x = \sqrt{3} \Rightarrow u = 4$.

$$\text{Then } \int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{1+x^2}} = \int_1^4 \frac{\frac{1}{2}du}{u^{1/2}} = \frac{1}{2}\int_1^4 u^{-1/2}du = u^{1/2}\Big]_1^4 = 2 - 1 = 1$$

17. D p. 102

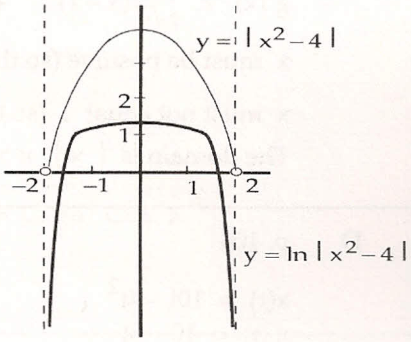
Solution I. Work analytically.
If $f(x) = \ln|x^2 - 4|$ on the interval $(-2,2)$,
then $f'(x) = \dfrac{2x}{x^2-4} = \dfrac{2x}{(x-2)(x+2)}$.

(A) $f'(x) < 0$ when $0 < x < 2$; f is decreasing then. **False.**
(B) $f(0) = \ln 4 \neq 0$, so $(0,0)$ is **not** on the graph. **False.**
(C) $|x^2 - 4|$ has a maximum value of 4 on the given domain, so $f(x)$ has a maximum value of $\ln(4)$. **False.**
(E) Since $f(0) = \ln 4$, f does **not** have an asymptote at $x = 0$. **False.**

Thus (D) must be **True**.

Solution II. Work graphically.
The graph of $y = |x^2 - 4|$, restricted to the open interval $(-2,2)$, is the tip of an upside-down parabola, as shown to the right.

$f(x) = \ln(y) = \ln|x^2 - 4|$ then has the darker graph shown. The selection of the correct answer is made easier with these graphs.



18. E p. 103

$g(x) = \text{Arcsin}(2x)$

$$g'(x) = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

Copyright 2006 Venture Publishing

19. B p. 103

$$\int x(x^2-1)^4\,dx = \frac{1}{2}\int (x^2-1)^4\,(2x\,dx)$$
$$= \frac{1}{2}\cdot\frac{(x^2-1)^5}{5} + C = \frac{1}{10}(x^2-1)^5 + C$$

20. B p. 103

$y = e^{kx}$ $\qquad \frac{d^3y}{dx^3} = k^3 e^{kx}$

$\frac{dy}{dx} = k e^{kx}$ $\qquad \frac{d^4y}{dx^4} = k^4 e^{kx}$

$\frac{d^2y}{dx^2} = k^2 e^{kx}$ $\qquad \frac{d^5y}{dx^5} = k^5 e^{kx}$

21. B p. 104

I. $f(2) = 1;\ f'(1) = 1.$ False

II. $\int_0^1 f(x)\,dx = -\frac{1}{2}$; $\quad f'(3.5) = -1$ True

III. $\int_{-1}^1 f(x)\,dx = -1$; $\quad \int_{-1}^2 f(x)\,dx = \frac{1}{2}$ False

22. C p. 104

$g(x) = \sqrt{x}\,(x-1)^{2/3}$

$g'(x) = \frac{1}{2\sqrt{x}}(x-1)^{2/3} + \frac{2}{3}(x-1)^{-1/3}\sqrt{x}$

x must be positive (so that $\frac{1}{2\sqrt{x}}$ exists).

x must not equal 1 (so that $(x-1)^{-1/3}$ exists).

The domain is $\{\, x \mid x > 0 \text{ and } x \neq 1 \,\} = \{\, x \mid 0 < x < 1 \text{ or } x > 1 \,\}$.

23. D p. 104

$x(t) = 10t - 4t^2$

$x'(t) = 10 - 8t$

$x'(t) > 0$ if $t < \frac{5}{4}$. The point is moving to the right.

$x'(t) < 0$ if $t > \frac{5}{4}$. The point is moving to the left.

The total distance traveled is $\left[x(\frac{5}{4}) - x(1)\right] + \left[x(\frac{5}{4}) - x(2)\right]$.

Since $x(\frac{5}{4}) = \frac{25}{4}$, $x(1) = 6$, and $x(2) = 4$, the total distance traveled is $\frac{5}{2}$.

Copyright 2006 Venture Publishing

24. D p. 105

$g'(x) = 0 \Rightarrow$ a horizontal tangent at x.
$g''(x) = 0 \Rightarrow$ no concavity (with possibly a change in concavity).

Only at points B, D, and E is there a horizontal tangent to the graph of g, so $g'(0) = 0$ at those points.

At B, the graph is concave up, so $g''(x) > 0$ there.
At D, the concavity changes, so $g''(x) = 0$ there.
At E, the graph is concave down, so $g''(x) < 0$ there.
Hence, point D is the answer.

25. B p. 105

$$xy - 2y + 4y^2 = 6$$
$$x\frac{dy}{dx} + y - 2\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-y}{x + 8y - 2}$$

When $y = 1$,
$x - 2 + 4 = 6$;
$\Rightarrow x = 4$.

Hence at the point (4,1), $\frac{dy}{dx} = \frac{-1}{4+8-2} = -\frac{1}{10}$

26. D p. 105

The area is given by $A = \int_0^{\pi/4} (1 + \sec^2 x)\,dx = [x + \tan x]_0^{\pi/4}$
$$= \frac{\pi}{4} + 1 \approx \frac{3}{4} + 1 = 1.75$$

27. A p. 106

Separate variables to solve this differential equation.

$$\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = -2xy$$
$$\Rightarrow \frac{dy}{y} = -2x\,dx$$
$$\Rightarrow \ln|y| = -x^2 + C$$

Since the curve contains the point (0, e), we have $\ln e = C$, so $C = 1$.
Thus $\ln y = -x^2 + 1$, or $y = e^{1-x^2}$.

28. A p. 106

I. $F(1) = \int_1^1 \ln(2t - 1)\,dt = 0$ since $\int_a^a f(t)\,dt = 0$ if f(a) exists. True

II. $F'(x) = \ln(2x - 1)$ by the Second Fundamental Theorem.
Then $F'(1) = \ln(1) = 0$. True

III. $F''(x) = \frac{2}{2x-1}$, so $F''(1) = 2$. False

Copyright 2006 Venture Publishing

Exam V
Section I
Part B — Calculators Permitted

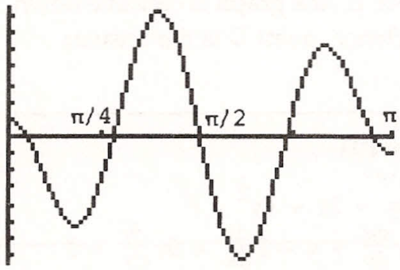
1. E p. 107

$y = \cos x + \frac{1}{3}\cos(3x) - \frac{1}{5}\cos(5x)$

$y' = -\sin x - \sin(3x) + \sin(5x)$

$y'' = -\cos x - 3\cos(3x) + 5\cos(5x)$

The graph to the right is the graph of y''. On the interval $[0,\pi]$, there are 5 places where the sign of y'' changes. Hence the graph of y has 5 points of inflection.



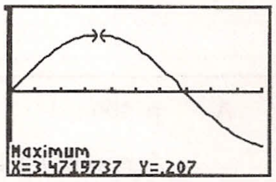
2. C p. 107

$$\text{Volume} = \int_0^{10} 500\, e^{-.2t}\, dt = \frac{500}{-.2}\left[e^{-.2t}\right]_0^{10} \approx 2161.7$$

3. B p. 108

Given the sales function $S(t)$, the rate of change of sales is given by $S'(t) = -.46\,(.45)\sin(.45t + 3.15)$.

This function is graphed to the right. The maximum value occurs when $t = 3.47$. This corresponds to a point in 1983.



4. E p. 108

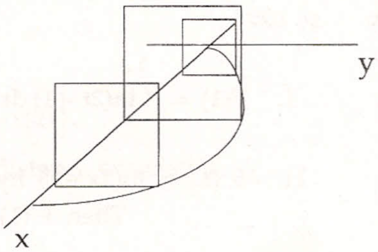
The midpoint approximation is given by

$M_4 = 2\,[\,f(2) + f(4) + f(6) + f(8)\,]$
$= 2\,[\,2 + 4 + 3 + 3\,] = 24$

5. D p. 108

The interval for one arch of the graph of $y = \sin x$ is $[0,\pi]$.
Since the regular cross sections are squares with edge $y = \sin x$, the volume of the solid is

$$V = \int_0^{\pi} (\sin x)^2\, dx \approx 1.57.$$



Copyright 2006 Venture Publishing

6. C p. 109

$$g(f(x)) = x \quad \Rightarrow \quad g'(f(x))\, f'(x) = 1$$
$$\Rightarrow \quad g'(f(x)) = \frac{1}{f'(x)}$$

To find $g'(2)$, first determine the number c such that $f(c) = 2$. Then we compute $\frac{1}{f'(c)}$.

To find the number c such that $f(c) = 2$, use your calculator to determine the zero of the function $F(x) = 2x + \sin x - 2$. It is $c = 0.684$.

Then $g'(f(.684)) = g'(2) = \frac{1}{f'(.684)} = \frac{1}{2 + \cos .684} = 0.36$.

---

7. D p. 109

$E(B) = 14000 + (B+1)^2$ and $B(t) = 20 \sin \frac{t}{10} + 50$.

$$\frac{dE}{dt} = \frac{dE}{dB} \cdot \frac{dB}{dt}$$
$$= 2(B+1) \cdot 2 \cos \frac{t}{10} = 4(B+1) \cos \frac{t}{10}$$

When $t = 100$, then $B = 20 \sin 10 + 50 \approx 39.12$.

Hence $\frac{dE}{dt} \approx -134.65$. Thus E is decreasing at $135 per day.

---

8. B p. 110

I.	f is increasing to the left of -1 and decreasing to the right.	True
II.	$f'(0) < 0$, so f is decreasing at $x = 0$.	False
III.	f' is decreasing on $(-2,0)$, so $f''(x) < 0$ on that interval.	True
IV.	f' has horizontal tangents at $x = 0, 2,$ and 3, and $f''(x)$ changes sign at each point.	True

---

9. E p. 110

To have the graph of f both concave down and increasing, we must have $f''(x) < 0$ <u>and</u> $f'(x) > 0$.

$$f(x) = 4x^{3/2} - 3x^2$$
$$f'(x) = 6x^{1/2} - 6x = 6x^{1/2}(1 - x^{1/2})$$
$$f''(x) = 3x^{-1/2} - 6 = 3x^{-1/2}(1 - 2x^{1/2})$$

Note that the presence of the factor $x^{-1/2}$ in $f''(x)$ forces $x > 0$, since $f''(x)$ must be defined in order to be negative.

Also $x^{1/2}$ is always positive, so it is the other factors we must deal with in determining whether $f'(x)$ and $f''(x)$ are positive or negative.

In order to have	$f'(x) > 0$	and	$f''(x) < 0$,
we must have	$1 - x^{1/2} > 0$	and	$1 - 2x^{1/2} < 0$.
Hence	$x^{1/2} < 1$	and	$\frac{1}{2} < x^{1/2}$.
Therefore	$0 < x < 1$	and	$\frac{1}{4} < x$

This interval is $(\frac{1}{4}, 1)$.

Copyright 2006 Venture Publishing

10. B p. 110

$$f(x) = x^2 - \frac{1}{e^x}$$

$$y_{\text{inst}} = f'(x) = 2x + \frac{1}{e^x}$$

$$y_{\text{ave}} = \frac{f(3) - f(0)}{3 - 0} = \frac{\left(9 - \frac{1}{e^3}\right)}{3} = \frac{10 - \frac{1}{e^3}}{3} = 3.3167$$

Solving $2x + \frac{1}{e^x} = 3.3167$ gives $x = 1.5525$.

---

11. E p. 111

Solution I. Differentiate the given equation: $3\cos(3x) = f(x)$.

$$\int_a^x 3\cos(3t)\,dt = \sin(3t)\Big]_a^x = \sin(3x) - \sin(3a) = \sin(3x) - 1.$$

Thus $\sin(3a) = 1$, so $3a = \frac{\pi}{2}$, and $a = \frac{\pi}{6}$.

Solution II. Let $x = a$, then $\sin 3a - 1 = \int_a^a f(t)\,dt = 0$.

Thus $\sin 3a = 1$, so $3a = \frac{\pi}{2}$, and $a = \frac{\pi}{6}$.

---

12. D p. 111

$xy^2 = 20$

Differentiate implicitly:

$$y^2\frac{dx}{dt} + 2xy\frac{dy}{dt} = 0$$

Since x is <u>decreasing</u> at a rate of 3 units/sec, we know that $\frac{dx}{dt} = -3$.

Also, when $y = 2$, then $x = 5$.

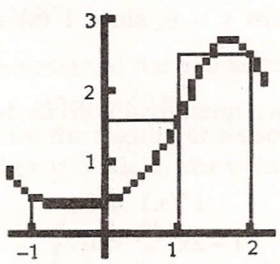
Substituting these values in the derivative equation yields

$$4(-3) + 2(5)(2)\frac{dy}{dt} = 0$$

Hence $20\frac{dy}{dt} = 12$ so $\frac{dy}{dt} = \frac{3}{5}$ units/sec.

---

13. C p. 111



$f(x) = e^{\sin(1.5x - 1)}$

We need a right-hand Riemann sum with three subintervals on the interval $[-1,2]$. The subintervals to be dealt with are: $[-1,0]$, $[0,1]$, and $[1,2]$.

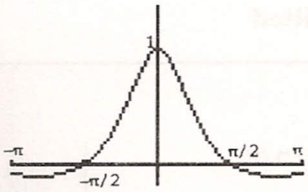
The right-hand function values are then:

$f(0) = 0.4311$
$f(1) = 1.6151$
$f(2) = 2.4826$

The sum of the area of the right-hand rectangles is then the sum of these three right-hand values (since each rectangle has width 1).

$R_3 = 0.4311 + 1.6151 + 2.4826 = 4.5288$

14. E p. 112



The graph of $\frac{dy}{dx} = \frac{\cos x}{x^2+1}$ shows two zeros (extrema for y) and three extrema (inflection points for y).

15. D p. 112

By the Second Fundamental Theorem, $G'(x) = f(x)$. Hence the given graph is the graph of the derivative of G.

I. $G'(x) = f(x) =< 0$ on $(1,2)$, so G is <u>decreasing</u> there. False
II. $G'(x) = f(x) = < 0$ on $(-4,-3)$, so G <u>is</u> decreasing there. True
III. $G(-4) = 0$ (since it is the integral from -4 to -4).
$G(x)$ is a decreasing function on the interval $[-4,0]$ (since its derivative is negative-valued there). Thus $G(0) < G(-4) = 0$ True

16. D p. 113

$\lim_{x\to1^-} f(x) = 1 + k - 3$ while $\lim_{x\to1^+} f(x) = 3 + b$.
For the function f to be continuous, these limits must be equal.
Hence $1 + k - 3 = 3 + b$; this simplifies to $k = 5 + b$.

For the function f to be differentiable, $\lim_{x\to1^-} f'(x)$ must equal $\lim_{x\to1^+} f'(x)$.
We find $\lim_{x\to1^-} f'(x) = \lim_{x\to1^-} (2x + k) = 2 + k$.
also $\lim_{x\to1^+} f'(x) = \lim_{x\to1^+} (3) = 3$.
Setting these last two limits equal, we have $2 + k = 3$, so $k = 1$.
Then from the previous condition ($k = 5 + b$), we have $b = -4$.

17. B p. 113

The position function is an antiderivative of the velocity function:
$v(t) = t + 2\sin t$.
Hence $x(t) = \frac{1}{2}t^2 - 2\cos t + C$.
The given condition that $x(0) = 0$ implies $0 = 0 - 2\cos 0 + C$.
Therefore $C = 2$, and the position function is $x(t) = \frac{1}{2}t^2 - 2\cos t + 2$.
Now we must determine the moment when the velocity is 6.
$v(t) = 6 \quad \Rightarrow \quad t + 2\sin t = 6$.
We solve this equation with our calculator: $t = 6.1887$.
The position at that moment is $x(6.1886965) \approx 19.1589$.

Copyright 2006 Venture Publishing

Exam V
Section II
Part A — Calculators Permitted

1. p. 115

(a) The average velocity over the interval $1 \le t \le 3$ is given by

$$\frac{x(3) - x(1)}{3 - 1} = \frac{(e^3 - \sqrt{3}) - (e^1 - \sqrt{1})}{2} = 8.318 \text{ ft/sec.}$$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

(b) The velocity function is given by

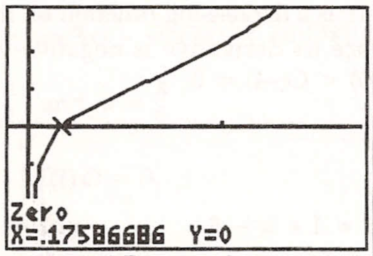
$$v(t) = x'(t) = e^t - \frac{1}{2\sqrt{t}}.$$

The velocity at time $t = 1$ is $v(1) = e - \frac{1}{2} = 2.218$ ft/sec.

Since the velocity at time $t = 1$ is positive, the particle is moving to the right (in the positive direction) on the x-axis at a speed of 2.218 ft/sec.

$3: \begin{cases} 1: \text{velocity } x'(t) \\ 2: \text{answer} \\ \quad 1: \text{direction} \\ \quad 1: \text{speed} \end{cases}$

(c) The particle is moving to the right when $v(t) = e^t - \frac{1}{2\sqrt{t}} > 0$. Shown to the right is a graph of the velocity function $v(t)$. When $t > 0.176$, the velocity is positive-valued. Hence the particle is moving to the right when $t > 0.176$ seconds.



$-.1 \le x \le 1.7 \quad -3.1 \le y \ge 3.1$

$2: \begin{cases} 1: \text{velocity } x'(t) > 0 \\ 1: \text{answer} \end{cases}$

(d) The velocity is 0 when $t = 0.176$ sec.
The particle's position at that moment is $x(0.176) = 0.773$ ft.

$2: \begin{cases} 1: \text{find } t \text{ when } x'(t) = 0 \\ 1: \text{answer} \end{cases}$

2. p. 116

(a) $L = 1 \cdot [4 + 5 + 7 + 11 + 12] = 39 \text{ m}^3.$
$U = 1 \cdot [5 + 7 + 11 + 12 + 14] = 49 \text{ m}^3.$

$2: \begin{cases} 1: \text{lower estimate} \\ 1: \text{upper estimate} \end{cases}$

(b) $\frac{L + U}{2} = 44 \text{ m}^3.$

Since the maximum total flow is 49 m^3 and the minimum possible flow is 39 m^3, this average is in error by <u>at most</u> 5 m^3.

$3: \begin{cases} 1: \text{average} \\ 1: \text{maximum error} \end{cases}$

(c)

$$F = \int_1^3 (t^2 - t + 5)\, dt = \left[\frac{t^3}{3} - \frac{t^2}{2} + 5t\right]_1^3$$

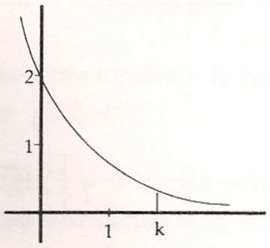
$$= \left[9 - \frac{9}{2} + 15\right] - \left[\frac{1}{3} - \frac{1}{2} + 5\right] = \frac{44}{3} = 14.667 \text{ m}^3$$

$4: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

Copyright 2006 Venture Publishing

3. p. 117

(a)



$$A = \int_0^k 2e^{-x}\,dx = \Big[-2e^{-x}\Big]_0^k = -2e^{-k} + 2 = 2 - \frac{2}{e^k}$$

$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(b) The volume of the solid revolved about the x-axis is done using the disk method.

$$V_x = \pi \int_0^k (2e^{-x})^2\,dx = \pi \int_0^k (4e^{-2x})\,dx$$

$$= \Big[-2\pi\, e^{-2x}\Big]_0^k$$

$$= -2\pi\,(e^{-2k} - 1) = 2\pi\left(1 - \frac{1}{e^{2k}}\right)$$

$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(c) $$\lim_{k\to\infty} V_x = \lim_{k\to\infty} 2\pi\left(1 - \frac{1}{e^{2k}}\right)$$

$$= 2\pi - 2\pi\left[\lim_{k\to\infty} \frac{1}{e^{2k}}\right]$$

$$= 2\pi - 2\pi \cdot 0 = 2\pi = 6.283$$

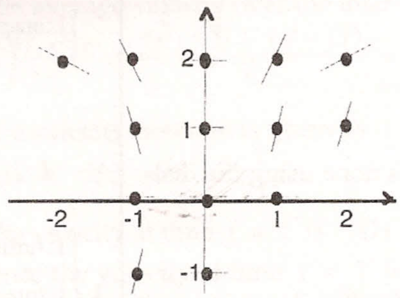
$3: \begin{cases} 1: \text{limit} \\ 2: \text{answer} \end{cases}$

Copyright 2006 Venture Publishing

Exam V
Section II
Part B — No Calculators

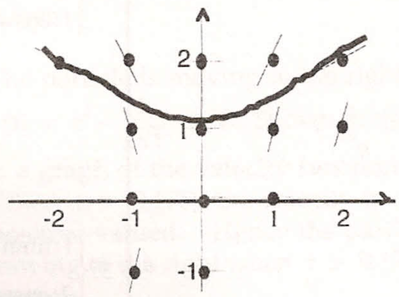
4. p. 118

(a)



$2: \begin{cases} 1: \text{zero slopes} \\ 1: \text{nonzero slopes} \end{cases}$

(b)



1 : solution curve mustgo through (-2,2), follow the given slope lines and extend to the border

(c) $\frac{dy}{dx} = \frac{xy}{x^2+4}$

We separate variables: $\frac{dy}{y} = \frac{x\,dx}{x^2+4}$.

Integrating both sides, we have $\ln|y| = \frac{1}{2}\ln|x^2+4| + C$.

This can be rewritten $\ln|y| = \ln\sqrt{x^2+4} + C$.

Taking exponentials of both sides yields

$$|y| = e^{\ln\sqrt{x^2+4}+C} = e^C \cdot e^{\ln\sqrt{x^2+4}}.$$

This gives $y = D\sqrt{x^2+4}$, where $D = \pm e^C$, depending upon the initial conditions on y.

$4: \begin{cases} 1: \text{separate variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{solve for } y \end{cases}$

(d) Substituting the point (0,4) into the general solution, we have $4 = D\sqrt{0+4}$.

From this we find that $D = 2$.

Hence the particular solution is $y = 2\sqrt{x^2+4}$.

$2: \begin{cases} 1: \text{solves for constant} \\ 1: \text{answer} \end{cases}$

Copyright 2000 Venture Publishing

5. p. 119

(a) $h(x) = f(g(x))$

$h'(x) = f'(g(x))\,g'(x)$

Then the function h has critical points when either $f'(g(x)) = 0$ or $g'(x) = 0$.

$f'(g(x)) = 0$ if $g(x) = 2$, which happens at $x = 1$ and $x = 4.5$.
$g'(x) = 0$ occurs at $x = 1$ and $x = 3$.

Thus h has three critical points: $x = 1, 3, 4.5$.

3: three critical points

(b) To help in this part of the problem, and also as an aid for part (d) which one might want to do first rather than last, we create a table of values of values of the function $h(x) = f(g(x))$.

x	h(x)	h′(x) = f′(g(x))·g′(x)
0	$f(g(0)) = f(5) = 0$	(−)(−) = +
1	$f(g(1)) = f(2) = 5$	(0)(0) = 0 rel max
2	$f(g(2)) = f(3) = 4$	(−)(+) = −
3	$f(g(3)) = f(4) = 3$	(−)(0) = 0 rel min
4	$f(g(4)) = f(3) = 4$	(−)(−) = +
4.5	$f(g(4.5)) = f(2) = 5$	(0)(−) = + rel max
5	$f(g(5)) = f(0) = 0$	(+)(−) = −

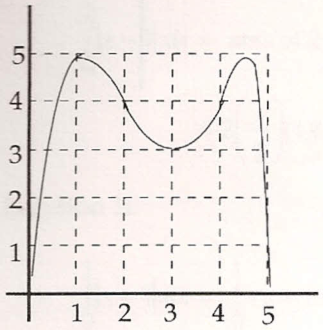
From the First Derivative Test, h has a local (relative) minimum at $x = 3$.
The function value there is 3. This gives the point (3, 3).

2: local minimum

(c) The function h is decreasing if $1 < x < 3$ or $4.5 < x < 5$.

2: intervals

(d)



2: graph

Copyright 2006 Venture Publishing

6. p. 120

(a) $f(x) = \ln\left[\frac{x}{x+1}\right]$

Since the domain of the natural logarithm function is the set of positive numbers, we must have $\frac{x}{x+1} > 0$.

This is true if and only if either $x > 0$ or $x < -1$.

Hence the domain of $f = \{\ x \mid x < -1 \text{ or } x > 0\}$.

$2: \begin{cases} 1: \frac{x}{x+1} > 0 \\ 1: \text{answer} \end{cases}$

(b) Since f(x) can be written $f(x) = \ln x - \ln(x+1)$ for $x > 0$,
and $f(x) = \ln|x| - \ln|x+1)|$ for $x < -1$, we have in either case

$$f'(x) = \frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}.$$

2: answer

(c) $f(1) = \ln\frac{1}{2} = -\ln 2$ while $f'(1) = \frac{1}{2}$.

Thus the tangent line at the point (1,f(1)) has the equation

$$y + \ln 2 = \frac{1}{2}(x-1).$$

$2: \begin{cases} 1: \text{slope} \\ 1: \text{tangent equation} \end{cases}$

(d) To find an expression for g(x), where g is the inverse of f, we interchange x and y in the rule for f(x), and solve for y.

$$\begin{aligned} x = \ln\left[\frac{y}{y+1}\right] \quad &\Rightarrow \quad e^x = \frac{y}{y+1} \\ &\Rightarrow \quad y e^x + e^x = y \\ &\Rightarrow \quad e^x = y - y e^x \\ &\Rightarrow \quad y(1-e^x) = e^x \\ &\Rightarrow \quad y = \frac{e^x}{1-e^x} \end{aligned}$$

Thus $g(x) = f^{-1}(x) = \frac{e^x}{1-e^x}$.

Then $g'(x) = \frac{(1-e^x)\cdot e^x - e^x \cdot (-e^x)}{(1-e^x)^2} = \frac{e^x}{(1-e)}$

$3: \begin{cases} 1: \text{interchange } x \text{ and } y \\ 1: \text{solve for } y \\ 1: \text{answer} \end{cases}$

Copyright 2006 Venture Publishing